*Journal of Statistical Physics, Vol. 54, Nos. 1/2, 1989*

# Linear Radiation Transport in Randomly Distributed Binary Mixtures: A One-Dimensional and Exact Treatment for the Scattering Case

**D. Vanderhaegen**[1] **and C. Deutsch**[2]

*Received May 10, 1988; revision received August 18, 1988*

Scattering effects are considered for radiative transfer within randomly distributed and binary mixtures in one dimension. The most general formalism is developed within the framework of the invariant imbedding method. The length $L$ of the random sample thus appears as a new variable. One transmission coefficient $T(L)$ suffices to specify locally the intensities. By analogy with the homogeneous situation, one introduces an effective opacity with $\langle T \rangle = (1 + \sigma_{\text{eff}} L)^{-1}$ fulfilling $\sigma_{\text{eff}} < \langle \sigma \rangle = p_0 \sigma_0 + p_1 \sigma_1$ (0 and 1 refer, respectively, to the components involved in the mixture). Equality is reached when $L \to 0, \infty$. Otherwise, $\sigma_{\text{eff}}$ displays a deep transmission window. It is numerically expressed for three combinations of opacities $(\sigma_0, \sigma_1)$ and average grain sizes $(\lambda_0, \lambda_1)$. These results are of crucial concern in optimizing an ICF compression for a pellet nonuniformly illuminated by intense laser or ion beams.

**KEY WORDS:** Random media; energy transport; Markov processes.

## 1. INTRODUCTION

Recently, an intense analytical and computational effort has been deployed[1-5] to estimate accurately the amount of radiative energy flowing through randomly distributed binary mixtures of various kinds of materials. On conceptual grounds, this could be partly appreciated as a testimony to the sustained interest into the venerable attenuation problem introduced in radar physics[6] and astrophysics.[7] However, it should be appreciated that it is only very recently[1-5] that a systematic interest has been devoted to understanding the radiation flow through randomly distributed mixtures. Plasma physicists engaged in laser or particule-

[1] CEN Limeil-Valenton, 94190, Villeneuve-Saint-Georges, France.
[2] LPGP (associé au CNRS), Université Paris XI, 91405, Orsay, France.



0022-4715/89/0100-0331$06.00/0 

beam-driven thermonuclear research[8,9] were indeed directly concerned. Although often quantitatively modest, the radiative energy balance plays a crucial role in assessing the credibility of currently investigated compression schemes. The importance of radiation transport in inertial confinement fusion (ICF) target calculations has been discussed many times.[9] It has been pointed out that radiation plays a much more important role in ion-beam-driven targets than in laser-fusion targets. This is because of the fact that in laser-fusion targets, suprathermal electron transport in general is more important than radiation transport and electron thermal conduction. In ion-beam fusion targets, on the other hand, radiation transport and the electron thermal conduction are the only means of energy transfer. The ions deposit energy in high-density target material, which makes electron thermal conduction very ineffective. In the absorption region of ion-beam targets the temperature increases to a few hundred eV. At these temperatures the radiation transport dominates the electron thermal conduction.

The presence of very different materials in the structure of fusion pellets produces Rayleigh–Taylor instabilites near the interfaces. These instabilities create mixing zones which are spatially localized. They can greatly influence radiative transfer and modify hydrodynamic compression.[9]

This type of heterogeneous mixture can be statistically described to derive the evolution of the mean radiative intensity and an effective opacity. As previously,[1-5] we shall focus attention on the basic and mathematical patterns involved in the stochastic differential equations, building up the backbone of the theoretical developments we are concerned with. The reader be able to transfer our results to other stochastic problems of interest.

We introduce a very simple frame (Fig. 1) for two media labeled as 0 and 1, respectively, interspersed at random, which we solve exactly in one dimension.

An instantaneous energy propagation is assumed. Moreover, we neglect the frequency dependence[6,7] (gray approximation). Many exact and accurate results have already been obtained[1-5] for the so-called attenuation problem. This means that we have a complete characterization for the attenuation at a given location $z$ within a random mix with prescribed averaged size grains $\lambda_0$ and $\lambda_1$, and also with opacity $\sigma_0$ and $\sigma_1$,

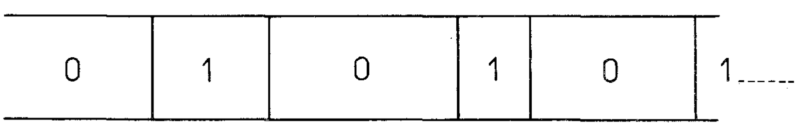


Fig. 1. Binary random mixture (medium 0 and medium 1) in one dimension.

respectively. In addition, each medium could be endowed with a specific temperature and emission coefficient. Technically speaking, the use of master equation methods[10,1] allows one to specify a prefactor in front of every exponential within a finite sum, while the attenuation coefficients are analytically derived from the stochastic parameters ($\lambda_0$, $\lambda_1$, $\sigma_0$, and $\sigma_1$).

The next step in generalizing this work[1-5] concerns the present paper. It is an attempt to incorporate scattering effects (back and forth) in the monodirectional model considered up to now. This extension stands as an obvious prerequisite before radiation transport theory can be realistically considered at higher dimensionality. We are thus considering two radiation intensities: $I^+(z)$ moving to the right, and $I^-(z)$ to the left. For physical reasons of rather immediate concern, we concentrate our attention on the special case where a given $L$ of random mix receives an initial energy density $i_+$ at $z=0$, while the extremity $z=L$ is exposed to $i_-$. Then, it turns out that the master equation approach[1] becomes useless because $I^+$ and $I^-$ have to be simultaneously prescribed at the left. This explains that $L$ should be taken as a new variable, a fact well documented by the so-called invariant imbedding method[11-13] (IIM) developed by many Russian workers. In so doing, we introduce an $L$-dependent transmission coefficient $T(L)$. A complete knowledge of $T(L)$ proves sufficient to determine local intensities $I^\pm(z, L)$ for all $z$ values. We are thus led to solve a stochastic and well-posed Cauchy problem for $T$. Its statistical distribution is worked out through a suitable master equation when the random mix is given the Markov property, the partial differential equation being solved through Green functions techniques. This allows us to specify the averages $\langle T \rangle$, $\langle T^{-1} \rangle$ for all $L$ values, together with an $L$-dependent effective opacity. The complete scattering problems is thus formally solved through $\langle I^\pm \rangle(z)$, locally specified.

The paper is organized as follows. Radiation transport including scattering is discussed within a one-dimensional framework in Section 2. The formal starting point for the whole paper is then given by Eqs. (9)–(11). The invariant imbedding method (IIM) is presented in Section 3 and Appendix A. The statistics of the transmission coefficient is worked out in Section 4, together with $\langle T \rangle$ and $\langle T^{-1} \rangle$. Green functions techniques required to solve the partial differential equations (63) are detailed in Appendix B. Exact analytical results are investigated in Section 5, while their numerical extension is worked out in Section 6.

## 2. RADIATION TRANSPORT WITH SCATTERING (1D)

It proves convenient to start with Sobolev's notations.[7] They are particularly suited for introducing the subject to a nonspecialized audience.

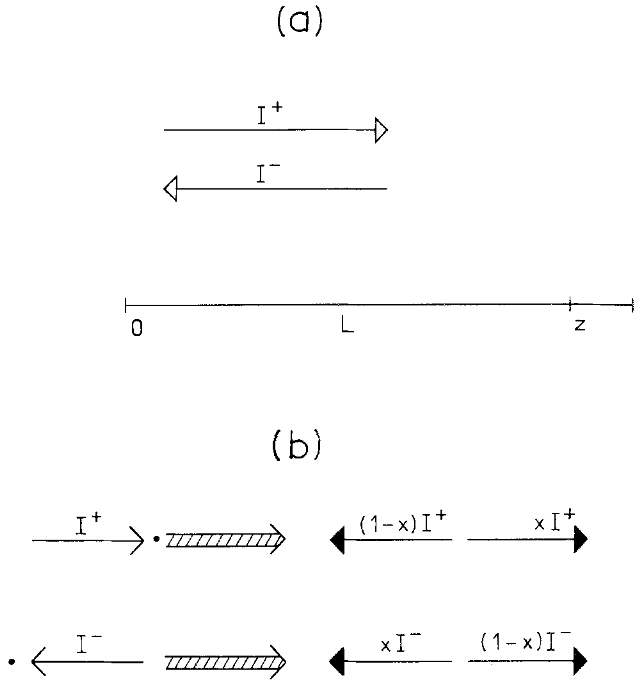


Fig. 1. Radiation transfer with scattering in one dimension (Sobolev notations). (a) Propagation without scattering; (b) propagation with scattering.

The relevant geometry is displayed in Fig. 2. It leads us to write a stationary radiative transport equation in the form

$$\begin{aligned} \frac{dI^+}{dz} &= -\sigma(z)I^+ + S^+ \\ -\frac{dI^-}{dz} &= -\sigma(z)I^- + S^- \end{aligned} \tag{1}$$

$\sigma(z)$ is the total opacity (attenuation) coefficient.

Source terms $S^\pm$ include the scattering coefficient and the thermal emission coefficient $S_T$. If $x$ denotes the proportion of scattered radiation, the source terms read

$$\begin{aligned} S^+ &= \Lambda[xI^+ + (1-x)I^-] + S_T^+ \\ S^- &= \Lambda[xI^- + (1-x)I^+] + S_T^- \end{aligned} \tag{2}$$

with $\Lambda$ the scattering coefficient. Thermal emission is given by

$$S_T = \sigma_{\text{abs}} B(T) = (\sigma - \Lambda)\, B(T) \tag{3}$$

$B(T)$ is the usual Planck function (the frequency dependence is omitted everywhere).

In a stationary regime and without any hydrodynamic motion in matter, one can put the energy balance in the form

$$\iint \sigma_{\mathrm{abs}}(B-I)\, d\mathbf{\Omega}\, d\nu = 0 \tag{4}$$

where the quadrature is taken over angles and frequency. In one dimension, $\mathbf{\Omega}$ is restricted to $\mathbf{\Omega} = \{-1, +1\}$, with

$$\int \sigma_{\mathrm{abs}}[2B - (I^+ + I^-)]\, d\nu = 0 \tag{5}$$

Restricting ourselves to a gray approximation, with $d\sigma_\nu/d\nu = 0$, we can use frequency-averaged quantities and write

$$2\sigma_{\mathrm{abs}} B = \sigma_{\mathrm{abs}}(I^+ + I^-) \tag{6}$$

Thus, Eq. (3) yields

$$S_T = (\sigma - \Lambda)(I^+ + I^-)/2 \tag{7}$$

Equations (2) and (7) finally lead to

$$\begin{aligned} S^+ &= [\sigma - \Lambda(1-2x)]I^+/2 + [\sigma + \Lambda(1-2x)]I^-/2 \\ S^- &= [\sigma + \Lambda(1-2x)]I^+/2 + [\sigma - \Lambda(1-2x)]I^-/2 \end{aligned} \tag{8}$$

Putting Eq. (8) into Eq. (1) provides the simple but fundamental system

$$\begin{aligned} \frac{dI^+}{dz} &= -\tilde{\sigma} I^+ + \tilde{\sigma} I^- \\ \frac{dI^-}{dz} &= -\tilde{\sigma} I^+ + \tilde{\sigma} I^- \end{aligned} \tag{9}$$

with

$$\tilde{\sigma} = [\sigma + \Lambda(1-2x)]/2 \tag{10}$$

$\tilde{\sigma}$ may be pictured as a two-state (0, 1) stochastic process with respect to the variable $z$. It corresponds to an absorption taking place either in 0 or 1.

Natural boundary conditions are

$$\begin{aligned} I^+(z=0) &= i_+ \\ I^-(z=L) &= i_- \end{aligned} \tag{11}$$

Equations (9)–(11) build up a formal starting point for the present work. A more physical insight may be gained by considering variables

$$\begin{aligned} E &= I^+ + I^- \qquad \text{(energy)} \\ F &= I^+ - I^- \qquad \text{(flux)} \end{aligned} \tag{12}$$

which allow us to express Eq. (9) as

$$\begin{aligned} \frac{dF}{dz} &= 0 \\ \frac{dE}{dz} &= -2\tilde{\sigma}F \end{aligned} \tag{13}$$

This corresponds to a conservative medium with a constant flux flowing throughout it. It is also instructive to consider a few degenerate situations.

First, let us consider $\tilde{\sigma}$ constant (no mix), so that

$$E = -2\tilde{\sigma}Fz + E_0 \tag{14}$$

Then if the light is incident on the left part only of the random mix with

$$i_+ = 1, \qquad i_- = 0 \tag{15}$$

one has

$$\begin{aligned} F &= (1+\tilde{\sigma}L)^{-1} \\ E_0 &= (1+2\tilde{\sigma}L)/(1+\tilde{\sigma}L) \end{aligned} \tag{16}$$

with reflection and transmission coefficients given, respectively, as

$$R = I^-(0), \qquad T = I^+(L) \tag{17}$$

For our conservative case

$$R + T = 1 \tag{18}$$

and for a constant $\tilde{\sigma}$

$$T = (1+\tilde{\sigma}L)^{-1} \tag{19}$$

## 3. INVARIANT IMBEDDING METHOD[11] (IIM)

Now, our main objective is to solve the system of Eqs. (9)–(11). It cannot be treated consistently through a master equation approach,[1,10] which requires the conditional probability

$$\mathscr{P}(I^+, I^-, \tilde{\sigma}, z \,|\, I^+, I^-, \tilde{\sigma}, z=0)$$

for the joint process $(I^+, I^-, \tilde{\sigma})$. The key point is that $I^+$ and $I^-$ are not simultaneously given at $z=0$. IIM gets rid of this difficulty by reformulating Eqs. (9)–(11) as a Cauchy problem. To simplify notation, $\tilde{\sigma}$ is now replaced by $\sigma$. The formal derivation, detailed in Appendix A, demonstrates that if $I^{\pm}$ is dependent on $(z, L)$ and also on the limit values $i_{\pm}$, respectively, taken at $z=0$ and $z=L$, with

$$I^{\pm} = I^{\pm}(z, L, i_+, i_-) \tag{20}$$

and the limit values

$$j^{\pm}(L, i_+, i_-) = I^{\pm}(L, L, i_+, i_-) \tag{21}$$

then one can recast Eqs. (9)–(11) in the form

$$\frac{\partial I^+}{\partial L} + (-\sigma j^+ + \sigma i_-)\frac{\partial I^-}{\partial i_-} = 0 \tag{22a}$$

$$\frac{\partial I^-}{\partial L} + (-\sigma j^+ + \sigma i_-)\frac{\partial I^-}{\partial i_-} = 0 \tag{22b}$$

with

$$I^+(z, z, i_+, i_-) = j^+(z, i_+, i_-) \tag{23a}$$

$$I^-(z, z, i_+, i_-) = j^-(z, i_+, i_-) = i_- \tag{23b}$$

and

$$\frac{\partial j^+}{\partial L} + (-\sigma j^+ + \sigma i_-)\frac{\partial j^+}{\partial i_-} = -\sigma j^+ + \sigma i_- \tag{24}$$

$$j^+(0, i_+, i_-) = i_+ \tag{25}$$

The linearity of Eq. (24) allows us to look for a solution in the form

$$j^+(L, i_+, i_-) = R(L)i_- + T(L)i_+ \tag{26}$$

so Eqs. (24), (25) may be rewritten as

$$\begin{aligned} \frac{dR}{dL} &= -\sigma(R-1)^2 \\ \frac{dT}{dL} &= \sigma(R-1)T \end{aligned} \tag{27}$$

with

$$R(0) = 0, \qquad T(0) = 1 \tag{28}$$

Equations (27)–(28) yield at once

$$\frac{dT}{dL} = -\sigma T^2 \tag{29}$$

where

$$T(0) = 1, \qquad T + R = 1 \tag{30}$$

It is also possible to solve Eqs. (22a), (22b) in the form

$$\begin{aligned} I^+ &= A^+(z, L) i_+ + B^+(z, L) i_- \\ I^- &= A^-(z, L) i_- + B^-(z, L) i_- \end{aligned} \tag{31}$$

For instance, Eqs. (22a), (23a) yield

$$\frac{dA^+}{dL} - \sigma T B^+ = 0, \qquad \frac{dB^+}{dL} + \sigma T B^+ = 0 \tag{32}$$

$$B^+(z, z) = 1 - T(z), \qquad A^+(z, z) = T(z) \tag{33}$$

With the factorization

$$B^+(z, L) = f^+(z)\, T(L) \tag{34}$$

this leads to

$$f^+(z) = [1 - T(z)]/T(z) \tag{35}$$

$$B^+(z, L) = \frac{T(L)[1 - T(z)]}{T(z)} \tag{36}$$

$$A^+(z, L) = 1 - B^+(z, L) \tag{37}$$

Thus, one can put Eq. (1) in the form

$$I^+ = \left\{1 - \frac{T(L)}{T(z)}[1 - T(z)]\right\} i_+ + \left\{\frac{T(L)}{T(z)}[1 - T(z)]\right\} i_- \tag{38}$$

$$I^- = \left[1 - \frac{T(L)}{T(z)}\right] i_+ + \frac{T(L)}{T(z)} i_- \tag{39}$$

with $I^+ - I^- = T(L)(i_+ - i_-)$. Thus $I^\pm$ is expressed in terms of a unique and macroscopic transmission coefficient $T(z)$ given as a solution of the differential equation (29) with stochastic coefficient $\sigma$. We are now faced with a Cauchy problem with prescribed initial conditions. Now, it is possible to investigate the statistical properties of $T$ through a master equation approach. It is readily checked that $I^{\pm}$ derived from Eqs. (38)–(39) with $T$ given by Eq. (29) are solutions of the system (9)–(11). It is also a rather easy matter to rederive Eqs. (38)–(39) in our conservative case. A straightforward left–right balance for the intensities considered at limit points $z=0$ and $z=L$ (see Fig. 3a) together with a similar one taken at the running point $z$ (Fig. 3b) allow us to express the outgoing intensities in terms of $T(z)$ as

$$I^{+}(z)=[1-T(z)]\,I^{-}(z)+T(z)i_{+} \tag{38'}$$

$$[1-T(L)]i_{+}+T(L)i_{-}=[1-T(z)]i_{+}+T(z)\,I^{-}(z) \tag{39'}$$

in a form equivalent to Eqs. (38)–(39). This approach implies a separate knowledge of $T(z)$, while it appears *sui generis* in the previous IIM, which can be extended to the nonconservative case with $R+T<1$ through two coupled equations for $R$ and $T$.

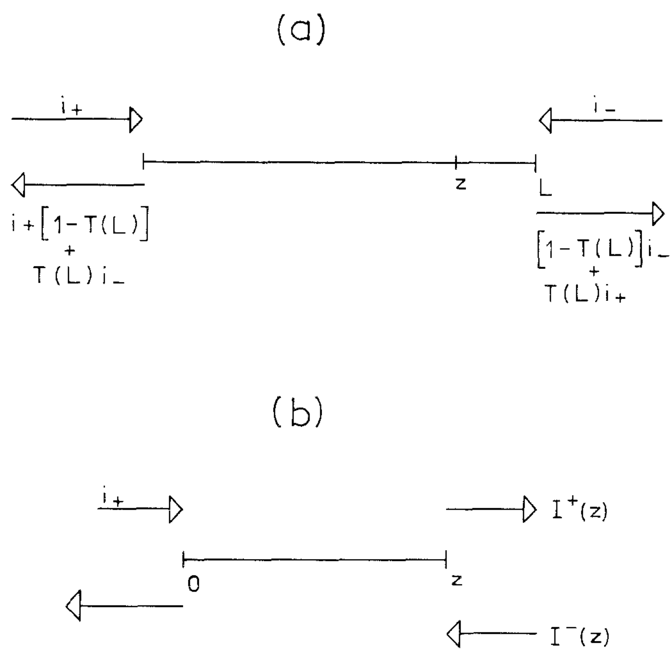


Fig. 3. Schematics of intensity balances involved in Eqs. (38′)–(39′). (a) At extremities $z=0$ and $L$, (b) at a running point $z$.

# 4. STATISTICS

## 4.1. Master Equation

Let us recall Eqs. (29)–(30),

$$\frac{dT}{dL} = -\sigma(L)T^2 \tag{40}$$

$$T(0) = 1 \tag{41}$$

where $\sigma$ denotes a two-state $(\sigma_0, \sigma_1)$ Markov process with characteristic lengths $\lambda_0$, $\lambda_1$. Then $(T, \sigma)$ is also Markovian, and we can write down a Liouville master equation[10] for the joint probability $\bar{\mathscr{P}}(T, \sigma, L)$,

$$\frac{\partial \bar{\mathscr{P}}}{\partial L} = -\frac{\partial}{\partial T}(-\bar{\bar{\sigma}} T^2 \bar{\mathscr{P}}) + \bar{\bar{W}} \bar{\mathscr{P}} \tag{42}$$

where

$$\bar{\mathscr{P}} - \begin{pmatrix} \mathscr{P}_0(T, L) = \mathscr{P}(T, \sigma = \sigma_0, L) \\ \mathscr{P}_1(T, L) = \mathscr{P}(T. \sigma = \sigma_1, L) \end{pmatrix} \tag{43}$$

and also

$$\bar{\bar{\sigma}} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \bar{\bar{W}} = \begin{pmatrix} -1/\lambda_0 & 1/\lambda_1 \\ 1/\lambda_0 & -1/\lambda_1 \end{pmatrix} \tag{44}$$

It remains to solve Eq. (42) with the initial $L = 0$ condition

$$\bar{\mathscr{P}}(T, L = 0) = \delta(T - 1) \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \tag{45}$$

$$p_i = \lambda_i(\lambda_0 + \lambda_1)^{-1}, \qquad i = 0, 1$$

and the limit constraint

$$\bar{\mathscr{P}}(T = 1, L) = 0, \qquad L = 0 \text{ excluded} \tag{46}$$

within a domain $\mathscr{D}$: $\{0 \leqslant T \leqslant 1,\ 0 \leqslant L \leqslant \infty\}$. Then, it is useful to express Eq. (42) in the form

$$\begin{aligned} \frac{\partial \mathscr{P}_0}{\partial L} &= \sigma_0 \frac{\partial}{\partial T}(T^2 \mathscr{P}_0) - \frac{\mathscr{P}_0}{\lambda_0} + \frac{\mathscr{P}_1}{\lambda_1} \\ \frac{\partial \mathscr{P}_1}{\partial L} &= \sigma_1 \frac{\partial}{\partial T}(T^2 \mathscr{P}_1) + \frac{\mathscr{P}_0}{\lambda_0} - \frac{\mathscr{P}_1}{\lambda_1} \end{aligned} \tag{47}$$

This is a coupled system of linear partial differential equations with non-constant coefficients, and with the problem (45), (46) split on the domain $\mathscr{D}$.

### 4.2. Canonical Formulation

Equation (47) is significantly simplified through the replacements

$$X = 1/T \tag{48}$$

$$\mathscr{P}_0 = X^2 Y_0(L, X), \qquad \mathscr{P}_1 = X^2 Y_1(L, X) \tag{49}$$

in the form

$$\begin{aligned} \frac{\partial Y_0}{\partial L} &= -\sigma_0 \frac{\partial Y_0}{\partial X} - \frac{Y_0}{\lambda_0} + \frac{Y_1}{\lambda_1} \\ \frac{\partial Y_1}{\partial L} &= -\sigma_1 \frac{\partial Y_1}{\partial X} + \frac{Y_0}{\lambda_0} - \frac{Y_1}{\lambda_1} \end{aligned} \tag{50}$$

with the limit conditions

$$\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} (X, L=0) = \delta(X-1) \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \tag{51}$$

$$\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} (X=1, L) = 0, \qquad L=0 \text{ excluded} \tag{52}$$

in $\mathscr{D}' = \{1 \leqslant X \leqslant \infty,\ 0 \leqslant L \leqslant \infty\}$.

Equations (50) may be further simplified with

$$Y_0 = e^{aX + AL} Y_0^*, \qquad Y_1 = e^{aX + AL} Y_1^* \tag{53}$$

and

$$a = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right)(\sigma_0 - \sigma_1) \tag{54}$$

$$A = \left(\frac{\sigma_1}{\lambda_0} - \frac{\sigma_0}{\lambda_1}\right)(\sigma_0 - \sigma_1) \tag{55}$$

Equations (50) now become

$$\begin{aligned} \frac{\partial Y_0^*}{\partial L} &= -\sigma_0 \frac{\partial Y_0^*}{\partial X} + \frac{Y_1^*}{\lambda_1} \\ \frac{\partial Y_1^*}{\partial L} &= -\sigma_1 \frac{\partial Y_1^*}{\partial X} + \frac{Y_0^*}{\lambda_0} \end{aligned} \tag{56}$$

while Eqs. (51) and (52) are replaced by

$$\begin{pmatrix} Y_0^* \\ Y_1^* \end{pmatrix} (X, L=0) = \delta(X-1) e^{-a} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \tag{57}$$

$$\begin{pmatrix} Y_0^* \\ Y_1^* \end{pmatrix} (X=1, L) = 0, \qquad L=0 \text{ excepted} \tag{58}$$

$Y_1^*$ may be further expressed in terms of $Y_0^*$, which yields two second-order and decoupled equations. Then, according to Sneddon,[14] it is possible to transform these hyperbolic equations into a simple canonical form by using

$$\xi = \gamma(X - 1 - \sigma_0 L), \qquad \eta = \gamma(X - 1 - \sigma_1 L) \tag{59}$$

$$\mathbf{Y}^*(X, L) = \tilde{\mathbf{y}}(\xi, \eta) \tag{60}$$

Equations (56) thus become

$$\begin{aligned} \frac{\partial \tilde{y}_0}{\partial \eta} &= \frac{\tilde{y}_1}{\lambda_1 \gamma(\sigma_0 - \sigma_1)} \\ \frac{\partial \tilde{y}_1}{\partial \xi} &= \frac{\tilde{y}_0}{\lambda_0 \gamma(\sigma_0 - \sigma_1)} \end{aligned} \tag{61}$$

where, if $\sigma_0 \geqslant \sigma_1$,

$$\gamma = 2(\lambda_0 \lambda_1)^{1/2} (\sigma_0 - \sigma_1) \tag{62}$$

Finally, we get the canonical equations

$$\begin{aligned} \frac{\partial^2 \tilde{y}_0}{\partial \xi\, \partial \eta} + \frac{\tilde{y}_0}{4} &= 0 \\ \frac{\partial^2 \tilde{y}_1}{\partial \xi\, \partial \eta} + \frac{\tilde{y}_1}{4} &= 0 \end{aligned} \tag{63}$$

### 4.3. Averages

The canonical expressions may be worked out with standard Green functions techniques detailed in Appendix B. This allows us to explain the statistical average of a given function $f(T)$ for a fixed $L$),

$$\begin{aligned} \langle f(T) \rangle_L &= \int_0^1 f(T)(\mathscr{P}_1 + \mathscr{P}_0)\, dT \\ &= \int_1^\infty f\left(\frac{1}{X}\right) [Y_0(L, X) + Y_1(L, X)]\, dX \end{aligned} \tag{64}$$

through Eqs. (48), (49), in the form [Eq. (B.13)]

$$\begin{aligned}\langle f(T)\rangle_L = {} & p_0 e^{-L/\lambda_0} f\left(\frac{1}{1+\sigma_0 L}\right) + p_1 e^{-L/\lambda_1} f\left(\frac{1}{1+\sigma_1 L}\right) \\ & + \frac{L}{(\lambda_0\lambda_1)^{1/2}} e^{-L/2\lambda_p} \int_{-\pi/2}^{\pi/2} d\Theta\, e^{L\frac{(\sin\Theta)}{2}(\lambda_1^{-1}-\lambda_0^{-1})} \\ & \times f\left[\frac{1}{1+(\sigma_0+\sigma_1)L/2+(\sigma_0-\sigma_1)L\sin\Theta/2}\right] \\ & \times \left\{\frac{I_1}{2}\left[\frac{L\cos\Theta}{(\lambda_0\lambda_1)^{1/2}}\right][1+(p_0-p_1)\sin\Theta] \right. \\ & \left. + \left(\frac{\lambda_0\lambda_1}{\lambda_0+\lambda_1}\right)^{1/2} I_0\left[\frac{L\cos\Theta}{(\lambda_0\lambda_1)^{1/2}}\right]\cos\Theta\right\}\end{aligned} \tag{65}$$

in terms of $\lambda_p^{-1} = \lambda_0^{-1} + \lambda_1^{-1}$. Here $I_0$ and $I_1$ are the standard modified Bessel functions of first kind with order $n = 0$ and 1, respectively.

## 5. EXACT RESULTS

### 5.1. $\langle T^{-1}\rangle$

Although $T$ statistics is somewhat tricky to access properly, the average $\langle T^{-1}\rangle$ is readily available from Eq. (29),

$$\frac{dT}{dL} = -\sigma(L)\,T^2$$

in the form

$$\frac{d(1/T)}{dL} = \sigma(L) \tag{66}$$

which may be treated by master equation approaches, through the replacement

$$u = T^{-1}$$

with

$$\frac{\partial \bar{\mathscr{P}}}{\partial L} = -\frac{\partial}{\partial u}(\bar{\bar{\sigma}}\bar{\mathscr{P}}) + \bar{\bar{W}}\bar{\mathscr{P}} \tag{67}$$

easily handled through the first moment

$$\bar{m}_u = \int u\bar{\mathscr{P}}\, du \tag{68}$$

in the form

$$\frac{\partial \bar{m}_u}{\partial L} = \bar{\bar{\sigma}} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} + \bar{\bar{W}}\bar{m}_u \tag{69}$$

as an intergration by parts. It reads

$$\begin{aligned} \frac{\partial m_0}{\partial L} &= \sigma_0 p_0 - \frac{m_0}{\lambda_0} + \frac{m_1}{\lambda_1} \\ \frac{\partial m_1}{\partial L} &= \sigma_1 p_1 + \frac{m_0}{\lambda_0} - \frac{m_1}{\lambda_1} \end{aligned} \tag{70}$$

with

$$m_0(0) = p_0, \qquad m_1(0) = p_1 \tag{70$'$}$$

Upon introducing

$$\langle u \rangle = m_0 + m_1 \tag{71}$$

one readily obtains

$$\frac{\partial \langle u \rangle}{\partial L} = \sigma_0 p_0 + \sigma_1 p_1 = \langle \sigma \rangle \tag{72}$$

Hence

$$\langle u \rangle = \langle \sigma \rangle L + 1 \tag{73}$$

i.e.,

$$\langle T^{-1} \rangle = 1 + \langle \sigma \rangle L \tag{73$'$}$$

which can also be obtained by direct averaging on Eq. (66).

Moreover, Schwarz' inequality gives

$$\langle T \rangle \langle T^{-1} \rangle \geqslant 1 \tag{74}$$

Then, let us introduce $\sigma_{\text{eff}}(L)$ according to Eq. (19), so that

$$\langle T \rangle = (1 - \sigma_{\text{eff}} L)^{-1} \tag{75}$$

Finally, Eqs. (73′) and (74) yield the inequality

$$\sigma_{\text{eff}}(L) \leqslant \langle \sigma \rangle, \qquad \text{all } L \tag{76}$$

## 5.2. $L \to 0$ Limit

It is obviously instructive to detail the $L=0$ limit of $\langle T \rangle_L$, in a first-order approximation with respect to $L$. So, we restrict Eq. (65) to its $L=0$, 1 terms, put $f(x)=x$, and obtain

$$\begin{aligned} \langle T \rangle_L &= p_0(1 - L/\lambda_0)(1 - \sigma_0 L) + p_1(1 - L/\lambda_0)(1 - \sigma_1 L) + 2/(\lambda_0 + \lambda_1) \\ &= 1 - L\langle \sigma \rangle, \qquad \langle \sigma \rangle = p_0 \sigma_0 + p_1 \sigma_1 \end{aligned} \tag{77}$$

Then Eq. (75) gives

$$\sigma_{\text{eff}} \to \langle \sigma \rangle, \qquad L \to 0 \tag{78}$$

in agreement with the expectation that the atomic mix model[3] should be retrieved as an $L \to 0$ limit.

## 5.3. $\langle T \rangle_L$ in the $L \to \infty$ Limit

Now we look for the dominant term $\sim O(L^{-1})$. Introducing the well-known $z \to 0$ limits

$$(2\pi z)^{1/2} I_0(z) \sim e^z, \qquad (2\pi z)^{1/2} I_1(z) \sim e^z \tag{79}$$

into Eq. (65) yields $[f(x)=x]$

$$\begin{aligned} \langle T \rangle_L \sim & \frac{p_0\, e^{-L/\lambda_0}}{1+\sigma_0 L} + p_1 \frac{e^{-L/\lambda_1}}{1+\sigma_1 L} \\ & + \frac{e^{-L/2\lambda_p}}{(\lambda_0 \lambda_1)^{1/2}} \int_{-\pi/2}^{\pi/2} d\Theta \exp\left\{ L \left[ \frac{\sin \Theta}{2} \left( \frac{1}{\lambda_1} - \frac{1}{\lambda_0} \right) + \frac{\cos \Theta}{(\lambda_0 \lambda_1)^{1/2}} \right] \right\} \\ & \times \left[ 1 + (p_0 - p_1) \frac{\sin \Theta}{2} + \frac{(\lambda_0 \lambda_1)^{1/2}}{\lambda_0 + \lambda_1} \cos \theta \right] (\lambda_0 \lambda_1)^{1/4} \\ & \times \left\{ \left[ \frac{\sigma_0 + \sigma_1}{2} + (\sigma_0 - \sigma_1) \frac{\sin \theta}{2} \right] (L^3 \cos \Theta)^{1/2} \right\}^{-1} \end{aligned} \tag{80}$$

which can be rewritten as

$$\frac{p_0 e^{-L/\lambda_0}}{1+\sigma_0 L} + \frac{p_1 e^{-L/\lambda_1}}{1+\sigma_1 L} + \frac{e^{-L/2\lambda_p}}{(\lambda_0 \lambda_1)^{1/4} (2\pi L)^{1/2}} J(L) \tag{81}$$

with

$$J(L)=\int_{-\pi/2}^{\pi/2} d\Theta\, f(\Theta)e^{L\Phi(\Theta)} \tag{82}$$

$$f(\Theta)=\frac{1+(p_0-p_1)\sin\Theta+2(\lambda_0\lambda_1)^{1/2}(\lambda_0+\lambda_1)^{-1}\cos\Theta}{2\cos^{1/2}\Theta\,[(\sigma_0+\sigma_1)/2+(\sigma_0-\sigma_1)(\sin\theta)/2]} \tag{83}$$

$$\Phi(\Theta)=\frac{\sin\Theta}{2}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_0}\right)+\frac{\cos\Theta}{(\lambda_0\lambda_1)^{1/2}} \tag{84}$$

By using Watson's lemma,[15] we derive the $L\to\infty$ limit of Laplace integrals, so that

$$J(L)\sim\frac{(2\pi)^{1/2}f(\Theta_0)e^{L\Phi(\Theta_0)}}{[-L\Phi''(\theta_0)]^{1/2}},\qquad L\to\infty \tag{85}$$

where $\Theta_0$ is such that $\Phi'(\Theta_0)=0$, where

$$\begin{gathered}
\operatorname{tg}\Theta_0=\frac{(\lambda_0\lambda_1)^{1/2}}{2}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_0}\right)\\
\cos\Theta_0=\frac{2\lambda_p}{(\lambda_0\lambda_1)^{1/2}},\qquad \sin\Theta_0=p_0-p_1\\
\Phi(\Theta_0)=-\Phi''(\Theta_0)=(2\lambda_p)^{-1}\\
f(\Theta_0)=\frac{(\lambda_0\lambda_1)^{1/4}}{(2\lambda_p)^{1/2}}\langle\sigma\rangle^{-1}
\end{gathered} \tag{86}$$

Equation (85) may be rewritten as

$$J(L)\sim\frac{(2\pi)^{1/2}(\lambda_0\lambda_1)^{1/4}\,e^{L/2\lambda_p}}{\langle\sigma\rangle\sqrt{L}} \tag{87}$$

which when introduced into Eq. (81) yields in the $L\to\infty$ limit

$$\langle T\rangle_L\sim\frac{1}{\langle\sigma\rangle L},\qquad L\to\infty \tag{88}$$

with [through Eq. (75)]

$$\sigma_{\text{eff}}\to\langle\sigma\rangle,\qquad L\to\infty \tag{89}$$

### 5.4. Symmetric Case $\lambda_0 = \lambda_1$ ($\sigma_0 = \sigma_1$)

Now, one has $p_0 = p_1 = 0.5$. It is easily checked that the distribution is a normalized one with $\langle 1 \rangle_L = 1$, all $L$. According to Eq. (65),

$$\langle 1 \rangle_L = e^{-l} + le^{-l} \int_0^{\pi/2} [I_1(l \cos \Theta) + I_0(l \cos \Theta) \cos \Theta]\, d\Theta, \qquad l = L/\lambda_0 \tag{90}$$

On the other hand, it is known that[16]

$$\int_0^{\pi/2} \cos(2\mu x)\, I_{2\nu}(2a \cos x)\, dx = \tfrac{1}{2}\pi I_{\nu-\mu}(a)\, I_{\nu+\mu}(a) \tag{91}$$

For $\nu = 1/2$, $\mu = 0$, $a = l/2$, one thus gets

$$\int_0^{\pi/2} I_1(l \cos \Theta)\, d\Theta = \tfrac{1}{2}\pi [I_{1/2}(l/2)]^2 \tag{92}$$

while for $\nu = 0$, $\mu = 1/2$, $a = l/2$, one obtains

$$\int_0^{\pi/2} I_0(l \cos \Theta) \cos \Theta\, d\Theta = \tfrac{1}{2}\pi I_{-1/2}(l/2)\, I_{1/2}(l/2) \tag{93}$$

The final result $\langle 1 \rangle_L = 1$ makes its appearance through

$$\begin{aligned} (2\pi z)^{1/2} I_{1/2}(z) &= e^z - e^{-z} \\ (2\pi z)^{1/2} I_{-1/2}(z) &= e^z + e^{-z} \end{aligned} \tag{94}$$

It is easily proved that

$$\langle T^{-1} \rangle = 1 + \langle \sigma \rangle L, \qquad \langle \sigma \rangle = (\sigma_0 + \sigma_1)/2 \tag{95}$$

### 5.5. $\langle I \rangle(z)$ Profile

Taking averages on Eqs. (38)–(39) yields at once

$$\langle I^+ \rangle(z) = \left[1 + \langle T(L) \rangle - \left\langle \frac{T(L)}{T(z)} \right\rangle\right] i_+ + \left[\left\langle \frac{T(L)}{T(z)} \right\rangle - \langle T(L) \rangle\right] i_- \tag{96}$$

$$\langle I^- \rangle(z) = \left[1 - \left\langle \frac{T(L)}{T(z)} \right\rangle\right] i_+ + \left\langle \frac{T(L)}{T(z)} \right\rangle i_- \tag{97}$$

To proceed further, one needs to explain $\langle T(L)/T(z)\rangle$. This could be achieved with

$$\left\langle \frac{T(L)}{T(z)} \right\rangle = \int_{T_0=1}^{T_0=0} \int_{T_1=T_0}^{T_1=0} \mathscr{P}(T(L)=T_1 \mid T(z)=T_0) \times \mathscr{P}(T(z)=T_0) \frac{T_1}{T_0}\, dT_1\, dT_0 \tag{98}$$

where

$$\mathscr{P}(T(L)=T_1 \mid T(z)=T_0) = \mathscr{P}_{T_{0,z}}(T_1, L)$$

is given by $\bar{\mathscr{P}}(T, L)$, the solution of the master equation

$$\frac{\partial \bar{\mathscr{P}}}{\partial L} = -\frac{\partial}{\partial T_1}(-\bar{\bar{\sigma}} T_1^2 \bar{\mathscr{P}}) + \bar{\bar{W}} \bar{\mathscr{P}} \tag{99}$$

with $(0 \leqslant T_1 \leqslant T_0,\ z \leqslant L \leqslant \infty)$ the limit value

$$\bar{\mathscr{P}} = \delta(T_1 - T_0) \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}, \qquad z = L$$

The variable transformation

$$\begin{aligned} T' &= T_1 T_0^{-1}, \qquad L' = T_0(L-z) \\ \bar{\bar{W}}' &= \bar{\bar{W}} T_0^{-1}, \qquad \bar{\mathscr{P}}' = \bar{\mathscr{P}} T_0 \end{aligned} \tag{100}$$

allows us to put Eq. (99) in the form

$$\begin{aligned} \frac{\partial \bar{\mathscr{P}}'}{\partial L'} &= -\frac{\partial}{\partial T'}(-\bar{\bar{\sigma}} T'^2 \bar{\mathscr{P}}) + \bar{\bar{W}}' \bar{\mathscr{P}} \\ \bar{\mathscr{P}}(L'=0, T') &= \delta(1-T') \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} \end{aligned} \tag{101}$$

already solved in Section 4.1. The knowledge of $\bar{\mathscr{P}}(T, z)$ is sufficient to formaly characterize $\langle I^+ \rangle(z)$ and $\langle I^- \rangle(z)$.

## 6. NUMERICAL RESULTS

A deeper exploration of the present formalism now requires a systematic numerical approach. For these purposes, $\langle 1 \rangle_L$, $\langle T^{-1} \rangle_L$, and $\langle T \rangle_L$ data are reported in Tables I–III as a function of $L$. They are

**Table I.** $\langle 1 \rangle_L$, $\langle T^{-1} \rangle_L$, and $\langle T \rangle_L$ deduced from Eq. (65) as well as $1 + \langle \sigma \rangle_L$ with $\sigma_0 = 9.0999$, $\sigma_1 = 0.1$, $\lambda_0 = 0.111$, $\lambda_1 = 1$, and $\lambda_p = 0.0999$

| $L$ | $\langle 1 \rangle_L$ | $\langle 1/T \rangle_L$ | $1 + \langle \sigma \rangle L$ | $\langle T \rangle$ |
|---|---|---|---|---|
| $20^{-4}$ | $1. + 10^{-7}$ | 1.00010 | 1.000099 | 0.9999 |
| $10^{-1}$ | $1. + 10^{-4}$ | 1.1001 | 1.0999 | 0.9363 |
| 0.5 | $1. + 2 \times 10^{-4}$ | 1.5006 | 1.4999 | 0.7776 |
| 1. | $1 + 9 \times 10^{-5}$ | 2.0006 | 1.9999 | 0.6305 |
| 2. | 0.999990 | 3.0001 | 2.9999 | 0.4393 |
| 5. | 0.999999 | 5.99998 | 5.99999 | 0.20741 |
| 10. | 0.999999 | 10.99999 | 10.99999 | 0.103767 |
| 20. | 0.999999 | 20.99998 | 20.99998 | 0.05108 |
| 50. | 0.999999 | 50.9999 | 50.99995 | 0.02018 |
| 100. | 0.999999 | 100.9999 | 100.9999 | 0.01004 |

parametrized by $\sigma_0$, $\sigma_1$, $p_0$, $p_1$, and $\lambda_p$ for specific cases already considered by Levermore *et al.*[4] in a related but different context. Under the assumption of inhomogeneous Markov statistics, these authors used the master equation formalism to derive the ensemble-averaged intensity and the distribution for the particle density as solutions of two coupled equations through an *ad hoc* approximation for the scattering processes in their rhs. For all of them, the expected relationships [Eq. (65)]

$$\begin{aligned} \langle 1 \rangle_L &= 1 \\ \langle T^{-1} \rangle_L &= 1 + \langle \sigma \rangle L \end{aligned} \tag{102}$$

are satisfied with a very high accuracy. Moreover, it is confirmed that $\sigma_{\text{eff}} \leqslant \langle \sigma \rangle$ with the equality fulfilled in the $L \to 0$ and $L \to \infty$ limits. $\sigma_{\text{eff}}$

**Table II.** $\langle 1 \rangle_L$, $\langle T^{-1} \rangle_L$, and $\langle T \rangle_L$ deduced from Eq. (65) as well as $1 + \langle \sigma \rangle_L$ with $\sigma_0 = 9.0999$, $\sigma_1 = 0.1$, $\lambda_0 = 1.1111$, $\lambda_1 = 10$, and $\lambda_p = 0.9999$

| $L$ | $\langle 1 \rangle_L$ | $\langle 1/T \rangle_L$ | $1 + \langle \sigma \rangle L$ | $\langle T \rangle$ |
|---|---|---|---|---|
| $10^{-4}$ | $1 + 10^{-8}$ | 1.0001 | 1.000999 | 0.9999 |
| $10^{-1}$ | $1 + 10^{-5}$ | 1.10002 | 1.0999 | 0.9426 |
| 0.5 | $1 + 7 \times 10^{-5}$ | 1.5002 | 1.49999 | 0.8588 |
| 1. | $1 + 10^{-4}$ | 2.0006 | 1.99999 | 0.787 |
| 2. | $1 + 2 \times 10^{-4}$ | 6.0049 | 5.9999 | 0.4374 |
| 5. | $1 + 9 \times 10^{-5}$ | 11.005 | 10.9999 | 0.2447 |
| 10. | 0.999990 | 21.002 | 20.9999 | 0.10500 |
| 20. | 0.99999 | 50.9999 | 50.9999 | 0.02822 |
| 50. | 0.9999997 | 100.99988 | 100.99989 | 0.011688 |
| 200. | $1.5 \times 10^{-11}$ | 200.99996 | 200.99996 | 0.0053767 |

**Table III.** $\langle 1\rangle_L$, $\langle T^{-1}\rangle_L$, and $\langle T\rangle_L$ deduced from Eq. (65) as well as $1+\langle\sigma\rangle_L$ with $\sigma_0=1.98$, $\sigma_1=0.02$, $\lambda_0=\lambda_1=5$, and $\lambda_p=2.5$

| $L$ | $\langle 1\rangle_L$ | $\langle 1/T\rangle_L$ | $1+\langle\sigma\rangle L$ | $\langle T\rangle$ |
|---|---|---|---|---|
| $10^{-4}$ | $1+10^{-8}$ | 1.000100 | 1.00001 | 0.99990 |
| $10^{-1}$ | $1+10^{-5}$ | 1.10001 | 1.1 | 0.91628 |
| 0.5 | $1+8\times10^{-5}$ | 1.5001 | 1.5 | 0.74119 |
| 1. | $1+10^{-4}$ | 2.0003 | 2. | 0.6379 |
| 2. | $1+3\times10^{-4}$ | 3.0008 | 3. | 0.52108 |
| 5. | $1.+6\times10^{-4}$ | 6.003 | 6. | 0.329 |
| 10. | $1+9\times10^{-4}$ | 11.0097 | 11. | 0.1752 |
| 20. | $1+10^{-3}$ | 21.027 | 21. | 0.07100 |
| 50. | $1+2\times10^{-3}$ | 51.1077 | 51. | 0.02210 |
| 100. | $1+3\times10^{-3}$ | 101.3 | 101. | 0.01046 |

exhibits a unique but asymmetric and bell-shaped minimum for intermediate $L$ values, which could be interpreted as a transmission window (see Figs. 4–6). This unexpected but potentially signifiant result for physics applications will be discussed more thoroughly in a forthcoming work.

The tabulated results as well as those displayed in Figs. 4–6 provide a demonstrative illustration of the above methods. They also pertain to three

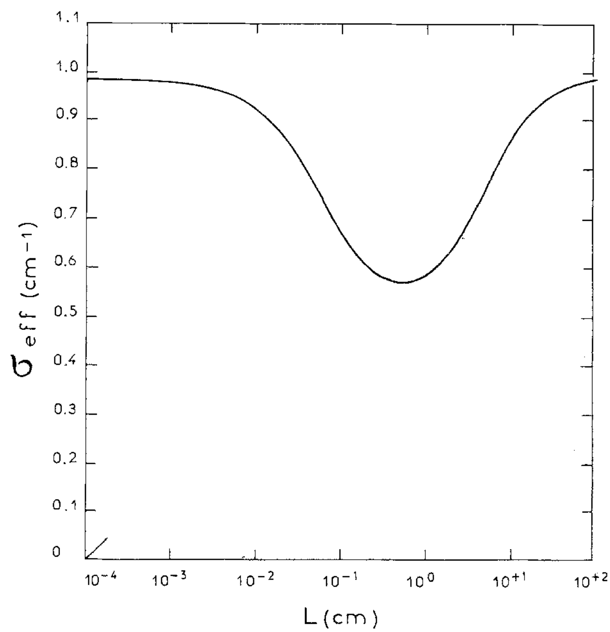


Fig. 4. $\sigma_{\rm eff}$ as a function of $L$ for parameters displayed in Table I.

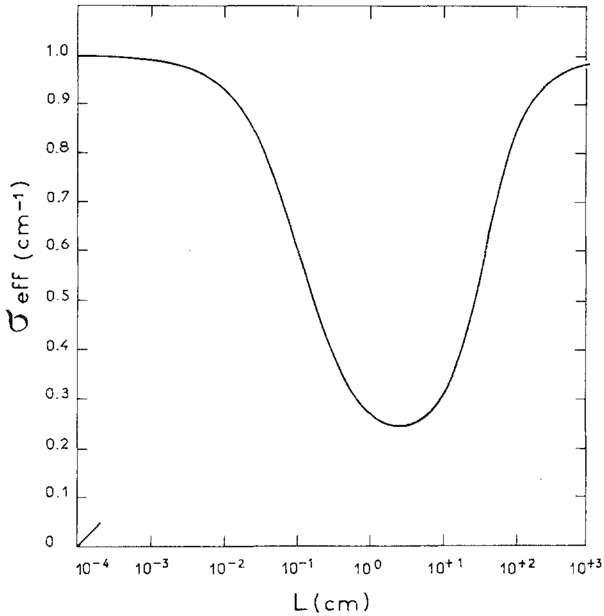


Fig. 5. Same as Fig. 4, for Table II.

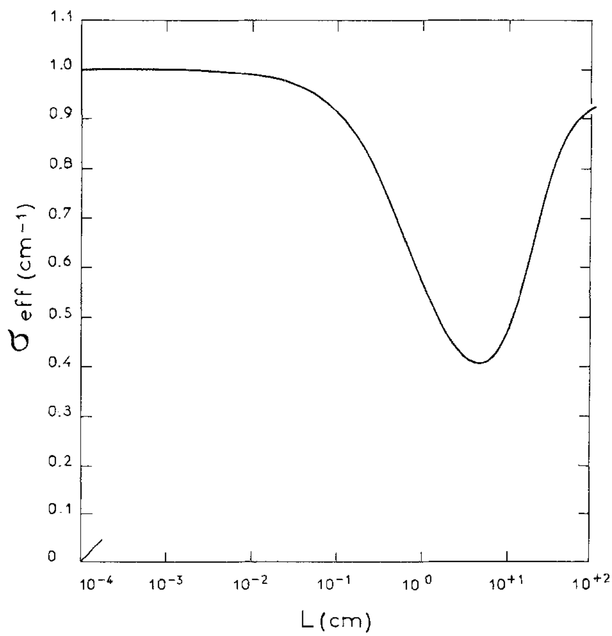


Fig. 6. Same as Fig. 4, for Table III.

sets of parameters ($\sigma_0$, $\sigma_1$, $\lambda_0$, $\lambda_1$) distinct enough to point out the numerical robustness of the corresponding averaging procedures. This is an important feature of the present approach in connection with its implementation within numerical codes devoted to the quantitative prediction of ICF compression toward a breakeven. In particular, the averaging techniques displayed here should allow one to work out a convenient procedure for correcting the so-called illumination nonuniformities arising from nonsymmetrical converging beams on a hollow target plasma.[17]

## APPENDIX A. INVARIANT IMBEDDING METHOD[11]

Let us consider a system of differential equations

$$\frac{dx_i}{dt} = F_i[t, \mathbf{x}(t)] \tag{A.1}$$

with limiting conditions in the form

$$g_{ik} x_k(0) + h_{ik} x_k(T) = v_i \tag{A.2}$$

where repeated indices are summed over. $\mathbf{x}(t)$ is now taken as $\mathbf{x}(t, T, \mathbf{v})$, i.e., a function of three variables. Now, let us differentiate with respect to $T$ and $v_k$, and invert the order of derivations. We thus obtain

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial x_i}{\partial T}(t, T, \mathbf{v})\right] &= \frac{\partial F_i}{\partial x_l}\frac{\partial x_l}{\partial T} \\ \frac{d}{dt}\left[\frac{\partial x_i}{\partial v_k}(t, T, \mathbf{v})\right] &= \frac{\partial F_i}{\partial x_l}\frac{\partial x_l}{\partial v_k} \end{aligned} \tag{A.3}$$

So, for every $i$, $\partial x_i/\partial T$ and $\partial x_i/\partial v_k$ satisfy the same differential equation with respect to $t$. There thus exists a linear relationship relating them, in the form

$$\frac{\partial x_i}{\partial T} = \lambda_k(T, \mathbf{v})\frac{\partial x_i}{\partial v_k} \tag{A.4}$$

where $\lambda_k(T, \mathbf{v})$ remains to be determined. At $t = 0$, Eq. (A.4) gives

$$\frac{\partial x_i}{\partial T}(0, T, \mathbf{v}) = \lambda_k(T, \mathbf{v})\frac{\partial x_i}{\partial v_k}(0, T, \mathbf{v}) \tag{A.5}$$

Multiplying Eq. (A.5) with $g_{li}$ (and summing over $i$), one gets

$$g_{li}\frac{\partial x_i}{\partial T}(0, T, \mathbf{v}) = g_{li}\lambda_k(T, \mathbf{v})\frac{\partial x_i}{\partial v_k}(0, T, \mathbf{v}) \tag{A.6}$$

Similarly, the same procedure applied at $t = T$ with $h_{li}$ yields

$$h_{li} \frac{\partial x_i}{\partial T}(t, T, \mathbf{v})_{t=T} = h_{li} \lambda_k(T, \mathbf{v}) \frac{\partial x_i}{\partial v_k}(t, T, \mathbf{v})_{t=T} \tag{A.7}$$

Adding (A.6) to (A.7) gives

$$\begin{aligned} g_{li} \frac{\partial x_i}{\partial T}(0, T, \mathbf{v}) + h_{li} \frac{\partial x_i}{\partial T}(t, T, \mathbf{v})_{t=T} \\ = \lambda_k(T, \mathbf{v}) \frac{\partial}{\partial v_k}[g_{li} x_i(0, T, \mathbf{v}) + h_{li} x_i(T, T, \mathbf{v})] \\ = \lambda_k(T, \mathbf{v}) \frac{\partial}{\partial v_k}(v_l) \end{aligned}$$

The last line arrived at through (A.2), and

$$g_{li} \frac{\partial x_i}{\partial T}(0, T, \mathbf{v}) + h_{li} \frac{\partial x_i}{\partial T}(t, T\mathbf{v}) = \lambda_l(T, \mathbf{v}) \tag{A.8}$$

One the other hand, Eq. (A.1) introduced into

$$\frac{d}{dT} x_k(T, T, \mathbf{v}) = \frac{\partial x_k}{\partial T}(t, T, \mathbf{v})_{t=T} + \frac{\partial x_k}{\partial t}(t, T, \mathbf{v})_{t=T} \tag{A.9}$$

yields

$$\frac{\partial x_k}{\partial T}(t, T, \mathbf{v})_{t=T} = \frac{dx_k}{dT}(T, T, \mathbf{v}) - F_k[T, \mathbf{x}(T, T, \mathbf{v})] \tag{A.10}$$

which, when inserted into (A.8), gives

$$\frac{d}{dT}[g_{li} x_i(0, T, \mathbf{v}) + h_{li} x_i(T, T, \mathbf{v})] - h_{li} F_i[T, \mathbf{x}(T, T, \mathbf{v})] = \lambda_l(T, \mathbf{v})$$

According to (A.2), the bracketed term on the lhs reduces to $v_l$, so that

$$\frac{d}{dT} v_l - h_{li} F_i[T, \mathbf{x}(T, T, \mathbf{v})] = \lambda_l(T, \mathbf{v})$$

i.e.,

$$\lambda_l(T, \mathbf{v}) = -h_{li} F_i[T, \mathbf{x}(T, T, \mathbf{v})] \tag{A.11}$$

Introducing (A.11) into (A.4) yields

$$\frac{\partial x_i}{\partial T}(t, T, \mathbf{v}) + h_{kl} F_l[T, \mathbf{x}(T, T, \mathbf{v})] \frac{\partial x_i}{\partial v_k}(t, T, \mathbf{v}) = 0 \tag{A.12}$$

which, upon the replacement

$$\mathbf{R}(T, \mathbf{v}) = \mathbf{x}(T, T, \mathbf{v}) \tag{A.13}$$

becomes

$$\frac{\partial x_i}{\partial T}(t, T, \mathbf{v}) + h_{kl} F_l[T, \mathbf{R}(T, \mathbf{v})] \frac{\partial x_i}{\partial v_k}(t, T, \mathbf{v}) = 0 \tag{A.14}$$

with also

$$x_i(t, t, \mathbf{v}) = R_i(t, \mathbf{v}) \tag{A.15}$$

Using again (A.9), one thus then derives from (A.14)

$$\begin{aligned} \frac{\partial x_i}{\partial T}(T, T, \mathbf{v}) &= \frac{\partial x_i}{\partial T}(t, T, \mathbf{v})\bigg|_{t=T} + \frac{\partial x_i}{\partial t}(t, T, \mathbf{v})\bigg|_{t=T} \\ &= -h_{kl} F_l[T, \mathbf{R}(T, \mathbf{v})] \frac{\partial x_i}{\partial v_k}(T, T, \mathbf{v}) + F_i[t, \mathbf{x}(T, T, \mathbf{v})] \end{aligned}$$

whence one obtains a partial differential equation for $\mathbf{R}$,

$$\frac{\partial R_i}{\partial T}(T, \mathbf{v}) + h_{kl} F_l[T, \mathbf{R}(T, \mathbf{v})] \frac{\partial R_i}{\partial v_k}(t, \mathbf{v}) = F_i[T, \mathbf{R}(T, \mathbf{v})] \tag{A.16}$$

with the limit requirements

$$R_i(0, \mathbf{v}) = x_i(t, T, \mathbf{v})|_{t=0, T=0}$$

For $T = 0$, (A.2) gives

$$g_{ik} R_k(0, \mathbf{v}) + h_{ik} R_k(0, \mathbf{v}) = v_i$$

hence

$$(g_{ik} + h_{ik}) R_k(0, \mathbf{v}) = v_i$$

and

$$\mathbf{R}(0, \mathbf{v}) = (g + h)^{-1} \mathbf{v} \tag{A.17}$$

Let us introduce

$$x_1 = I^+, \qquad x_2 = I^-, \qquad t = z, \qquad T = L \tag{A.18}$$

Then Eq. (9) of the text is written

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}[t, \mathbf{x}(t)] \tag{A.19}$$

where

$$F_1 = -\sigma x_1 + \sigma x_2, \qquad F_2 = -\sigma x_1 + \sigma x_2$$

Moreover, condition (11) may be given the form

$$\bar{\bar{g}}\mathbf{x}(0) + \bar{\bar{h}}\mathbf{x}(T) = \mathbf{v} \tag{A.20}$$

with

$$\bar{\bar{g}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \bar{\bar{h}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.21}$$

and

$$v_1 = i_+, \qquad v_2 = i_- \tag{A.22}$$

Let us also define

$$\begin{aligned} j^+(L, i_+, i_-) &= I^+(L, L, i_+, i_-) \\ j^-(L, i_+, i_-) &= I^-(L, L, i_+, i_-) \end{aligned} \tag{A.23}$$

Then Eqs. (A.14)–(A.15) may be transcribed as

$$\begin{aligned} \frac{\partial I^+}{\partial L} + (-\sigma j^+ + \sigma j^-)\frac{\partial I^+}{\partial i_-} &= 0 \\ \frac{\partial I^-}{\partial L} + (-\sigma j^+ + \sigma j^-)\frac{\partial I^-}{\partial i_-} &= 0 \end{aligned} \tag{A.24}$$

where

$$\begin{aligned} j^+(z, i_+, i_-) &= I^+(z, z, i_+, i_-) \\ j^-(z, i_+, i_-) &= I^-(z, z, i_+, i_-) \end{aligned} \tag{A.25}$$

Similarly, Eqs. (16)–(17) become

$$\frac{\partial j^+}{\partial L}+(-\sigma j^+ +\sigma j^-)\frac{\partial j^+}{\partial i_-}= -\sigma j^+ +\sigma j^- \tag{A.26a}$$

$$\frac{\partial j^-}{\partial L}+(-\sigma j^+ +\sigma j^-)\frac{\partial j^+}{\partial i_-}= -\sigma j^+ +\sigma j^- \tag{A.26b}$$

with

$$j^+(0, i_+, i_-)=i_+, \qquad j^+(0, i_+, i_-)=i_- \tag{A.27}$$

Moreover, Eqs. (A.20) and (A.23) lead to

$$j^-(L, i_+, i_-)=i_- \tag{A.28}$$

Finally, the system (A.24), (A.25), (A.26a), and (A.27) may be summarized as

$$\begin{aligned} &\frac{\partial I^+}{\partial L}+(-\sigma j^+ +\sigma i_-)\frac{\partial I^+}{\partial i_-}=0\\ &\frac{\partial I^-}{\partial L}+(-\sigma j^+ +\sigma i_-)\frac{\partial I^-}{\partial i_-}=0 \end{aligned} \tag{A.29}$$

with

$$\begin{aligned} &I^+(L=z, z, i_+, i_-)=j^+(z, i_+, i_-)\\ &I^-(L=z, z, i_+, i_-)=i_- \end{aligned} \tag{A.30}$$

and

$$\frac{\partial j^+}{\partial L}+(-\sigma j^+ +\sigma i_-)\frac{\partial j^+}{\partial i_-}= -\sigma j^+ +\sigma i_- \tag{A.31}$$

$$j^+(0, i_+, i_-)=i_+ \tag{A.32}$$

while (A.26b) reduces to a tautology.

## APPENDIX B. SOLUTION OF EQS. (63) WITH GREEN FUNCTION METHODS

Let us recall Eqs. (63),

$$\begin{aligned} &\frac{\partial^2 \tilde{y}_0}{\partial\xi\,\partial\eta}+\frac{\tilde{y}_0}{4}=0\\ &\frac{\partial^2 \tilde{y}_1}{\partial\xi\,\partial\eta}+\frac{\tilde{y}_1}{4}=0 \end{aligned} \tag{B.1}$$

Then the Green function techniques (see, for instance, Sneddon[14]) allow us to solve Eqs. (B1) with the limiting conditions (57), (58) transcribed in the $(\xi, \eta)$ variables accoding to Fig. 7. For a given location $P(\xi_0, \eta_0)$ one thus obtains

$$\begin{aligned}\tilde{y}_0(P) &= [w\tilde{y}_0]_B - \int_{AB} \left( \tilde{y}_0 \frac{\partial w}{\partial \xi} d\xi + w \frac{\partial \tilde{y}_0}{\partial \eta} d\eta \right) \\ \tilde{y}_1(P) &= [w\tilde{y}_1]_A - \int_{AB} \left( \tilde{y}_1 \frac{\partial w}{\partial \eta} d\eta + w \frac{\partial \tilde{y}_1}{\partial \xi} d\xi \right)\end{aligned} \tag{B.2}$$

where $w(\xi, \eta, \xi_0, \eta_0)$ is a Green function defined by

$$\begin{aligned}&\frac{\partial^2 w}{\partial \xi\, \partial \eta} + \frac{w}{4} = 0 \\ &\frac{\partial w}{\partial \xi} = 0, \qquad \eta = \eta_0 \\ &\frac{\partial w}{\partial \eta} = 0, \qquad \xi = \xi_0 \\ &w = 1, \qquad \xi = \xi_0, \qquad \eta = \eta_0\end{aligned} \tag{B.3}$$

When $P$ is located in zones 1 or 3 (Fig. 7), one thus gets $\tilde{y}_0 = \tilde{y}_1 = 0$. On the other hand, when $P$ belongs to zone 2 with $\xi_0 \leqslant 0$ and $\eta_0 \geqslant 0$, one has

$$\begin{aligned}\tilde{y}_0(\xi_0, \eta_0) &= w(\xi = 0, \eta = 0, \xi_0 = 0, \eta_0)\, p_0\, \delta(X - 1 - \sigma_0 L) e^{-a} \\ &\quad - \int_{AB} \left[ p_0\, \delta(X-1) e^{-a} \frac{\partial w}{\partial \xi} (\xi = \eta, \xi_0, \eta_0)\, d\xi \right. \\ &\quad \left. + w(\xi = \eta, \xi_0, \eta_0)\, p_1 \frac{\delta(X-1) e^{-a}}{\gamma \lambda_1 (\sigma^0 - \sigma_1)} d\eta \right]\end{aligned} \tag{B.4}$$

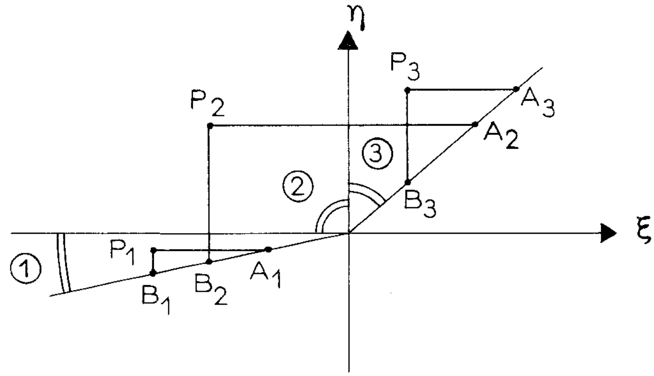


Fig. 7. Domains of definition in the $(\xi, \eta)$ plane for the Green function $w$ used for the solution of Eqs. (63) [Eq. (B.1)].

$$\tilde{y}_0(\xi_0, \eta_0) = w(\xi = 0, \eta = 0, \xi_0, \eta_0 = 0)\, p_1 \delta(X - 1 - \sigma_1 L) e^{-a}$$
$$+ \int_{AB} \left[ p_1\, \delta(X-1) e^{-a} \frac{\partial w}{\partial \eta} \left( \xi_0, \eta_0, \frac{\xi}{\sigma_0} = \frac{\eta}{\sigma_1} \right) d\eta \right.$$
$$\left. + w\left( \xi_0, \eta_0, \frac{\xi}{\sigma_0} = \frac{\eta}{\sigma_1} \right) p_0 \frac{\delta(X-1) e^{-a}}{\gamma \lambda_0 (\sigma_0 - \sigma_1)} d\xi \right] \tag{B.5}$$

If we notice that

$$\delta(X-1)\, dX = -\gamma \int_{AB} \delta[\gamma(X-1)]\, dX \tag{B.6}$$

Eqs. (B.4) and (B.5) become ($\xi_0 \leqslant 0$, $\eta_0 \geqslant 0$)

$$\tilde{y}_0(\xi_0, \eta_0) = p_0\, \delta(X - 1 - \sigma_0 L) e^{-a}$$
$$+ \gamma \left[ p_0 e^{-a} \frac{\partial w}{\partial \xi} (\xi = 0, \eta = 0, \xi_0, \eta_0) \right.$$
$$\left. + p_1 \frac{z^{-a} w(0, 0, \xi_0, \eta_0)}{\gamma \lambda_1 (\sigma_0 - \sigma_1)} \right] \tag{B.7}$$
$$\tilde{y}_1(\xi_0, \eta_0) = p_1\, \delta(X - 1 - \sigma_1 L) e^{-a}$$
$$- \gamma \left[ p_1 e^{-a} \frac{\partial w}{\partial \eta} (0, 0, \xi_0, \eta_0) \right.$$
$$\left. - p_0 \frac{e^{-a} w(0, 0, \xi_0, \eta_0)}{\gamma \lambda_0 (\sigma_0 - \sigma_1)} \right]$$

where

$$w(\xi, \eta, \xi_0, \eta_0) = I_0[\sqrt{(\eta_0 - \eta)(\xi - \xi_0)}] \tag{B.8}$$

explains the Green function in terms of the usual modified Bessel function of first kind and order $n = 0$. Working backward from the relationships (59), one retrieves the initial variables with

$$Y_0(X, L) = p_0 e^{-L/\lambda_0}\, \delta(X - 1 - \sigma_0 L)$$
$$+ \gamma e^{a(X-1) + AL} \left[ p_0 \frac{I_1}{2} \left( \frac{\eta_0}{-\xi_0} \right)^{1/2} + \frac{p_1 I_0}{\gamma \lambda_1 (\sigma_0 - \sigma_1)} \right] \tag{B.9}$$
$$Y_1(X, L) = p_1 e^{-L/\lambda_1}\, \delta(X - 1 - \sigma_1 L)$$
$$+ \gamma e^{a(X-1) + AL} \left[ p_1 \frac{I_1}{2} \left( \frac{-\xi_0}{\eta_0} \right)^{1/2} + \frac{p_1 I_0}{\gamma \lambda_0 (\sigma_0 - \sigma_1)} \right]$$

and

$$I_0 = I_0((-\xi_0\eta_0)^{1/2}), \qquad I_1 = I_0'((-\xi_0\eta_0)^{1/2}) \tag{B.10}$$

$$\xi_0 = \gamma(X - 1 - \sigma_0 L), \qquad \eta_0 = \gamma(X - 1 - \sigma_1 L) \tag{B.11}$$

so that

$$1 + \sigma_1 L \leqslant X \leqslant 1 + \sigma_0 L \tag{B.12}$$

On the other hand, $Y_0 = Y_1 = 0$, when $X < 1 + \sigma_1 L$ or $X > 1 + \sigma_0 L$.

If one recalls that $X = T^{-1}$, Eq. (B.12) becomes

$$(1 + \sigma_0 L)^{-1} \leqslant T \leqslant (1 + \sigma_1 L)^{-1} \tag{B.12$'$}$$

with bounds given by a unique homogeneous medium (0 and 1, respectively). The introduction of Eq. (B.9) into the second line of Eq. (64) gives

$$\begin{aligned}\langle f(T)\rangle_L = {} & p_0\, e^{-L/\lambda_0} f\left(\frac{1}{1+\sigma_0 L}\right) + p_1\, e^{-L/\lambda_1}\left(\frac{1}{1+\sigma_1 L}\right) \\ & + \gamma e^{AL} \int_{1+\sigma_1 L}^{1+\sigma_0 L} dX\, e^{a(X-1)} f(X^{-1}) \\ & \times \left\{\frac{I_1}{2}\left[p_0\left(\frac{\eta_0}{-\xi_0}\right)^{1/2} + p_1\left(\frac{-\xi_0}{\eta_0\cdot}\right)^{1/2}\right]\right. \\ & \left. + \frac{2I_0}{\gamma(\sigma_0 - \sigma_1)(\lambda_0 + \lambda_1)}\right\}\end{aligned} \tag{B.13}$$

where $\xi_0$, $\eta_0$, $A$, $a$, $I_0$, $I$, and $\gamma$ are defined by Eqs. (B.11), (54), (55), (B.10), and (62) respectively. Equation (B.13) may be simplified further through

$$\sin\Theta = [X - 1 - (\sigma_0 + \sigma_1)L/2][(\sigma_0 - \sigma_1)L/2]^{-1} \tag{B.14}$$

which yields Eq. (65) in the text.

## ACKNOWLEDGMENTS

It is a pleasure to thank P. Fromy for very efficient numerical help with $\langle 1\rangle_L = 1$. We are also deeply indebted to A. Munier for suggesting several crucial steps used in Section 4.2.

## REFERENCES


1. D. Vanderhaegen, *J. Quant. Spectrosc. Rad. Transfer* **36**:557 (1986).
2. D. Vanderhaegen, *J. Quant. Spectrosc. Rad. Transfer*, to be published.
3. C. D. Levermore, G. C. Pomraning, D. L. Sanzo, and J. Wong, *J. Math. Phys.* **27**:2526 (1986).
4. C. D. Levermore, G. C. Pomraning, and J. Wong, *J. Math. Phys.*, to be published.
5. D. Vanderhaegen, in preparation.
6. A. I. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978).
7. V. V. Sobolev, *A Treatise on Radiative Transfer* (Van Nostrand, Princeton, New Jersey, 1963), p. 31.
8. H. Y. Kull, *Phys. Rev. A* **33**:1957 (1986).
9. C. Deutsch, *Laser Part. Beams* **2**:449 (1984); *Ann. Phys. Fr.* **11**:1 (1986).
10. R. Kubo, *J. Math. Phys.* **4**:174 (1963).
11. V. I. Klyatshkin, *Waves and Stochastic Equations in Random and Inhomogeneous Media* (Editions de Physique, Orsay, 1986).
12. R. Rammal and B. Doucot, *J. Phys.* (Paris) **48**:509 (1987).
13. B. Doucot and R. Rammal, *J. Phys.* (Paris) **48**:527 (1987).
14. I. N. Sneddon, *Elements of Partial Differential Equations* (McGraw-Hill, New York).
15. C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).
16. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1965).
17. C. Deutsch, P. Fromy, X. Garbet, and G. Maynard, *Fus. Tech.* **13**:362 (1988).